

Calculus Made Easy

What one fool can do, another can.

(Ancient Simian Proverb.)

Calculus Made Easy:

*being a very-simplest introduction to those
beautiful methods of reckoning which are
generally called by the terrifying names of the*

DIFFERENTIAL CALCULUS

and the

INTEGRAL CALCULUS.

by

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Second Edition, Enlarged

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Preface to the Second Edition

THE surprising success of this work has led the author to add a considerable number of worked examples and exercises. Advantage has also been taken to enlarge certain parts where experience showed that further explanations would be useful.

The author acknowledges with gratitude many valuable suggestions and letters received from teachers, students, and—critics.

October, 1914.

Prologue

CONSIDERING how many fools can calculate, it is surprising that it should be thought either a difficult or a tedious task for any other fool to learn how to master the same tricks.

Some calculus-tricks are quite easy. Some are enormously difficult. The fools who write the textbooks of advanced mathematics—and they are mostly clever fools—seldom take the trouble to show you how easy the easy calculations are. On the contrary, they seem to desire to impress you with their tremendous cleverness by going about it in the most difficult way.

Being myself a remarkably stupid fellow, I have had to unteach myself the difficulties, and now beg to present to my fellow fools the parts that are not hard. Master these thoroughly, and the rest will follow. What one fool can do, another can.

To deliver you from the Preliminary Terrors

THE preliminary terror, which chokes off most fifth-form boys from even attempting to learn how to calculate, can be abolished once for all by simply stating what is the meaning—in common-sense terms—of the two principal symbols that are used in calculating.

These dreadful symbols are:

(1) d which merely means “a little bit of.”

Thus dx means a little bit of x ; or du means a little bit of u . Ordinary mathematicians think it more polite to say “an element of,” instead of “a little bit of.” Just as you please. But you will find that these little bits (or elements) may be considered to be indefinitely small.

(2) $\int$ which is merely a long S , and may be called (if you like) “the sum of.”

Thus $\int dx$ means the sum of all the little bits of x ; or $\int dt$ means the sum of all the little bits of t . Ordinary mathematicians call this symbol “the integral of.” Now any fool can see that if x is considered as made up of a lot of little bits, each of which is called dx , if you add them all up together you get the sum of all the dx ’s, (which is the same thing as the whole of x). The word “integral” simply means “the whole.” If you think of the duration of time for one hour, you may (if you like) think of it as cut up into 3600 little bits called seconds. The whole of the 3600 little bits added up together make one hour.

When you see an expression that begins with this terrifying symbol, you will henceforth know that it is put there merely to give you instructions that you are now to perform the operation (if you can) of totalling up all the little bits that are indicated by the symbols that follow.

That’s all.

I

d

∫

II

CHAPTER

On Different Degrees of Smallness

WE shall find that in our processes of calculation we have to deal with small quantities of various degrees of smallness.

We shall have also to learn under what circumstances we may consider small quantities to be so minute that we may omit them from consideration. Everything depends upon relative minuteness.

Before we fix any rules let us think of some familiar cases. There are 60 minutes in the hour, 24 hours in the day, 7 days in the week. There are therefore 1440 minutes in the day and 10080 minutes in the week.

Obviously 1 minute is a very small quantity of time compared with a whole week. Indeed, our forefathers considered it small as compared with an hour, and called it “one minùte,” meaning a minute fraction—namely one sixtieth—of an hour. When they came to require still smaller subdivisions of time, they divided each minute into 60 still smaller parts, which, in Queen Elizabeth’s days, they called “second minùtes” (*i.e.* small quantities of the second order of minuteness). Nowadays we call these small quantities of the second order of smallness “seconds.” But few people know *why* they are so called.

Now if one minute is so small as compared with a whole day, how much smaller by comparison is one second!

Again, think of a farthing as compared with a sovereign: it is barely worth more than $\frac{1}{1000}$ part. A farthing more or less is of precious little importance compared with a sovereign: it may certainly be regarded as a *small* quantity. But compare a farthing with £1000: relatively to this greater sum, the farthing is of no more importance than $\frac{1}{1000}$ of a farthing would be to a sovereign. Even a golden sovereign is relatively a negligible quantity in the wealth of a millionaire.

Now if we fix upon any numerical fraction as constituting the proportion which for any purpose we call relatively small, we can easily state other fractions of a higher degree of smallness. Thus if, for the purpose of time, $\frac{1}{60}$ be called a *small* fraction, then $\frac{1}{60}$ of $\frac{1}{60}$ (being a *small* fraction of a *small* fraction) may be regarded as a *small quantity of the second order* of smallness.*

Or, if for any purpose we were to take 1 per cent. (*i.e.* $\frac{1}{100}$) as a *small* fraction, then 1 per cent. of 1 per cent. (*i.e.* $\frac{1}{10,000}$) would be a small fraction of the second order of smallness; and $\frac{1}{1,000,000}$ would be a small fraction of the third order of smallness, being 1 per cent. of 1 per cent. of 1 per cent.

Lastly, suppose that for some very precise purpose we should regard $\frac{1}{1,000,000}$ as “small.” Thus, if a first-rate chronometer is not to lose or gain more than half a minute in a year, it must keep time with an accuracy of 1 part in 1,051,200. Now if, for such a purpose, we regard $\frac{1}{1,000,000}$ (or one millionth) as a small quantity, then $\frac{1}{1,000,000}$ of $\frac{1}{1,000,000}$, that is $\frac{1}{1,000,000,000,000}$ (or one trillionth) will be a small quantity of the second order of smallness, and may be utterly disregarded, by comparison.

Then we see that the smaller a small quantity itself is, the more negligible does the corresponding small quantity of the second order become. Hence we know that *in all cases we are justified in neglecting the small quantities of the second—or third (or higher)—orders*, if only we take the small quantity of the first order small enough in itself.

But, it must be remembered, that small quantities if they occur in our expressions as factors multiplied by some other factor, may become important if the other factor is itself large. Even a farthing becomes important if only it is multiplied by a few hundred.

Now in the calculus we write dx for a little bit of x . These things such as dx , and du , and dy , are called “differentials,” the differential of x , or of u , or of y , as the case may be. [You *read* them as *dee-eks*, or *dee-you*, or *dee-wy*.] If dx be a small bit of x , and relatively small of itself, it does not follow that such quantities as $x \cdot dx$, or $x^2 dx$, or $a^x dx$ are negligible. But $dx \times dx$ would be negligible, being a small quantity of the second order.

A very simple example will serve as illustration.

Let us think of x as a quantity that can grow by a small amount so as to become $x + dx$, where dx is the small increment added by

* The mathematicians talk about the second order of “magnitude” (*i.e.* greatness) when they really mean second order of *smallness*. This is very confusing to beginners.

growth. The square of this is $x^2 + 2x \cdot dx + (dx)^2$. The second term is not negligible because it is a first-order quantity; while the third term is of the second order of smallness, being a bit of, a bit of x^2 . Thus if we took dx to mean numerically, say, $\frac{1}{60}$ of x , then the second term would be $\frac{2}{60}$ of x^2 , whereas the third term would be $\frac{1}{3600}$ of x^2 . This last term is clearly less important than the second. But if we go further and take dx to mean only $\frac{1}{1000}$ of x , then the second term will be $\frac{2}{1000}$ of x^2 , while the third term will be only $\frac{1}{1,000,000}$ of x^2 .

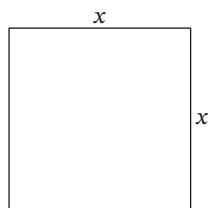


Fig. 1

Geometrically this may be depicted as follows: Draw a square (Fig. 1) the side of which we will take to represent x . Now suppose the square to grow by having a bit dx added to its size each way. The enlarged square is made up of the original square x^2 , the two rectangles at the top and on the right, each of which is of area $x \cdot dx$ (or together $2x \cdot dx$), and the little square at the top right-hand corner which is $(dx)^2$. In Fig. 2 we have taken dx as

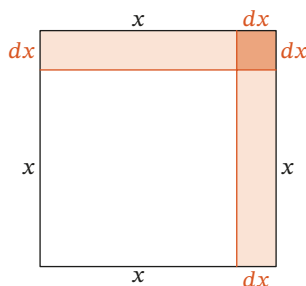


Fig. 2

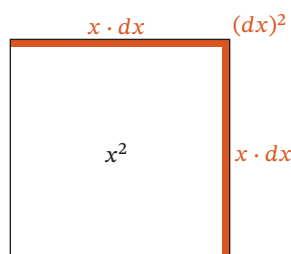


Fig. 3

quite a big fraction of x —about $\frac{1}{5}$. But suppose we had taken it only $\frac{1}{100}$ —about the thickness of an inked line drawn with a fine pen. Then the little corner square will have an area of only $\frac{1}{10,000}$ of x^2 , and be practically invisible. Clearly $(dx)^2$ is negligible if only we consider the increment dx to be itself small enough.

Let us consider a simile.

Suppose a millionaire were to say to his secretary: next week I will give you a small fraction of any money that comes in to me. Suppose that the secretary were to say to his boy: I will give you a small fraction of what I get. Suppose the fraction in each case to be $\frac{1}{100}$ part. Now if Mr. Millionaire received during the next week £1000, the secretary would receive £10 and the boy 2 shillings. Ten pounds would be a small quantity compared with £1000; but two shillings is a small small quantity indeed, of a very secondary order. But what

would be the disproportion if the fraction, instead of being $\frac{1}{100}$, had been settled at $\frac{1}{1000}$ part? Then, while Mr. Millionaire got his £1000, Mr. Secretary would get only £1, and the boy less than one farthing!

The witty Dean Swift* once wrote:

“So, Nat’ralists observe, a Flea
 “Hath smaller Fleas that on him prey.
 “And these have smaller Fleas to bite ’em,
 “And so proceed *ad infinitum*.”

* *On Poetry: a Rhapsody*
 (p. 20), printed
 1733—usually misquoted.

An ox might worry about a flea of ordinary size—a small creature of the first order of smallness. But he would probably not trouble himself about a flea’s flea; being of the second order of smallness, it would be negligible. Even a gross of fleas’ fleas would not be of much account to the ox.

III

CHAPTER

On Relative Growings

ALL through the calculus we are dealing with quantities that are growing, and with rates of growth. We classify all quantities into two classes: *constants* and *variables*. Those which we regard as of fixed value, and call *constants*, we generally denote algebraically by letters from the beginning of the alphabet, such as a , b , or c ; while those which we consider as capable of growing, or (as mathematicians say) of “varying,” we denote by letters from the end of the alphabet, such as x , y , z , u , v , w , or sometimes t .

Moreover, we are usually dealing with more than one variable at once, and thinking of the way in which one variable depends on the other: for instance, we think of the way in which the height reached by a projectile depends on the time of attaining that height. Or we are asked to consider a rectangle of given area, and to enquire how any increase in the length of it will compel a corresponding decrease in the breadth of it. Or we think of the way in which any variation in the slope of a ladder will cause the height that it reaches, to vary.

Suppose we have got two such variables that depend one on the other. An alteration in one will bring about an alteration in the other, *because* of this dependence. Let us call one of the variables x , and the other that depends on it y .

Suppose we make x to vary, that is to say, we either alter it or imagine it to be altered, by adding to it a bit which we call dx . We are thus causing x to become $x + dx$. Then, because x has been altered, y will have altered also, and will have become $y + dy$. Here the bit dy may be in some cases positive, in others negative; and it won’t (except by a miracle) be the same size as dx .

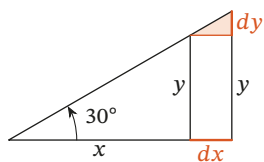


Fig. 4

Take two examples.

(1) Let x and y be respectively the base and the height of a right-angled triangle (Fig. 4), of which the slope of the other side is fixed

at 30° . If we suppose this triangle to expand and yet keep its angles the same as at first, then, when the base grows so as to become $x + dx$, the height becomes $y + dy$. Here, increasing x results in an increase of y . The little triangle, the height of which is dy , and the base of which is dx , is similar to the original triangle; and it is obvious that the value of the ratio $\frac{dy}{dx}$ is the same as that of the ratio $\frac{y}{x}$. As the angle is 30° it will be seen that here

$$\frac{dy}{dx} = \frac{1}{1.73}.$$

(2) Let x represent, in Fig. 5, the horizontal distance, from a wall, of the bottom end of a ladder, AB , of fixed length; and let y be the height it reaches up the wall. Now y clearly depends on x . It is easy to see that, if we pull the bottom end A a bit further from the wall, the top end B will come down a little lower. Let us state this in scientific language. If we increase x to $x + dx$, then y will become $y - dy$; that is, when x receives a positive increment, the increment which results to y is negative.

Yes, but how much? Suppose the ladder was so long that when the bottom end A was 19 inches from the wall the top end B reached just 15 feet from the ground. Now, if you were to pull the bottom end out 1 inch more, how much would the top end come down? Put it all into inches: $x = 19$ inches, $y = 180$ inches. Now the increment of x which we call dx , is 1 inch: or $x + dx = 20$ inches.

How much will y be diminished? The new height will be $y - dy$. If we work out the height by Euclid I. 47, then we shall be able to find how much dy will be. The length of the ladder is

$$\sqrt{(180)^2 + (19)^2} = 181 \text{ inches.}$$

Clearly then, the new height, which is $y - dy$, will be such that

$$(y - dy)^2 = (181)^2 - (20)^2 = 32761 - 400 = 32361,$$

$$y - dy = \sqrt{32361} = 179.89 \text{ inches.}$$

Now y is 180, so that dy is $180 - 179.89 = 0.11$ inch.

So we see that making dx an increase of 1 inch has resulted in making dy a decrease of 0.11 inch.

And the ratio of dy to dx may be stated thus:

$$\frac{dy}{dx} = -\frac{0.11}{1}.$$

3.1

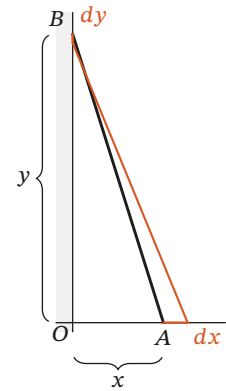


Fig. 5

3.2

3.3

3.4

It is also easy to see that (except in one particular position) dy will be of a different size from dx .

Now right through the differential calculus we are hunting, hunting, hunting for a curious thing, a mere ratio, namely, the proportion which dy bears to dx when both of them are indefinitely small.

It should be noted here that we can only find this ratio $\frac{dy}{dx}$ when y and x are related to each other in some way, so that whenever x varies y does vary also. For instance, in the first example just taken, if the base x of the triangle be made longer, the height y of the triangle becomes greater also, and in the second example, if the distance x of the foot of the ladder from the wall be made to increase, the height y reached by the ladder decreases in a corresponding manner, slowly at first, but more and more rapidly as x becomes greater. In these cases the relation between x and y is perfectly definite, it can be expressed mathematically, being $\frac{y}{x} = \tan 30^\circ$ and $x^2 + y^2 = l^2$ (where l is the length of the ladder) respectively, and $\frac{dy}{dx}$ has the meaning we found in each case.

If, while x is, as before, the distance of the foot of the ladder from the wall, y is, instead of the height reached, the horizontal length of the wall, or the number of bricks in it, or the number of years since it was built, any change in x would naturally cause no change whatever in y ; in this case $\frac{dy}{dx}$ has no meaning whatever, and it is not possible to find an expression for it. Whenever we use differentials dx , dy , dz , etc., the existence of some kind of relation between x , y , z , etc., is implied, and this relation is called a “function” in x , y , z , etc.; the two expressions given above, for instance, namely $\frac{y}{x} = \tan 30^\circ$ and $x^2 + y^2 = l^2$, are functions of x and y . Such expressions contain implicitly (that is, contain without distinctly showing it) the means of expressing either x in terms of y or y in terms of x , and for this reason they are called *implicit functions* in x and y ; they can be respectively put into the forms

$$\begin{aligned} y &= x \tan 30^\circ \quad \text{or} \quad x = \frac{y}{\tan 30^\circ} \\ \text{3.5} \quad \text{and} \quad y &= \sqrt{l^2 - x^2} \quad \text{or} \quad x = \sqrt{l^2 - y^2}. \end{aligned}$$

These last expressions state explicitly (that is, distinctly) the value of x in terms of y , or of y in terms of x , and they are for this reason called *explicit functions* of x or y . For example $x^2 + 3 = 2y - 7$ is an implicit function in x and y ; it may be written $y = \frac{x^2+10}{2}$ (explicit

function of x) or $x = \sqrt{2y - 10}$ (explicit function of y). We see that an explicit function in x , y , z , etc., is simply something the value of which changes when x , y , z , etc., are changing, either one at the time or several together. Because of this, the value of the explicit function is called the *dependent variable*, as it depends on the value of the other variable quantities in the function; these other variables are called the *independent variables* because their value is not determined from the value assumed by the function. For example, if $u = x^2 \sin \theta$, x and θ are the independent variables, and u is the dependent variable.

Sometimes the exact relation between several quantities x , y , z either is not known or it is not convenient to state it; it is only known, or convenient to state, that there is some sort of relation between these variables, so that one cannot alter either x or y or z singly without affecting the other quantities; the existence of a function in x , y , z is then indicated by the notation $F(x, y, z)$ (implicit function) or by $x = F(y, z)$, $y = F(x, z)$ or $z = F(x, y)$ (explicit function). Sometimes the letter f or ϕ is used instead of F , so that $y = F(x)$, $y = f(x)$ and $y = \phi(x)$ all mean the same thing, namely, that the value of y depends on the value of x in some way which is not stated.

We call the ratio $\frac{dy}{dx}$ "the *differential coefficient* of y with respect to x ." It is a solemn scientific name for this very simple thing. But we are not going to be frightened by solemn names, when the things themselves are so easy. Instead of being frightened we will simply pronounce a brief curse on the stupidity of giving long crack-jaw names; and, having relieved our minds, will go on to the simple thing itself, namely the ratio $\frac{dy}{dx}$.

In ordinary algebra which you learned at school, you were always hunting after some unknown quantity which you called x or y ; or sometimes there were two unknown quantities to be hunted for simultaneously. You have now to learn to go hunting in a new way; the fox being now neither x nor y . Instead of this you have to hunt for this curious cub called $\frac{dy}{dx}$. The process of finding the value of $\frac{dy}{dx}$ is called "differentiating." But, remember, what is wanted is the value of this ratio when both dy and dx are themselves indefinitely small. The true value of the differential coefficient is that to which it approximates in the limiting case when each of them is considered as infinitesimally minute.

Let us now learn how to go in quest of $\frac{dy}{dx}$.

NOTE TO CHAPTER III.

How to read Differentials.

It will never do to fall into the schoolboy error of thinking that dx means d times x , for d is not a factor—it means “an element of” or “a bit of” whatever follows. One reads dx thus: “dee-eks.”

In case the reader has no one to guide him in such matters it may here be simply said that one reads differential coefficients in the following way. The differential coefficient

$\frac{dy}{dx}$ is read “*dee-wy by dee-eks,*” or “*dee-wy over dee-eks.*”

So also $\frac{du}{dt}$ is read “*dee-you by dee-tee.*”

Second differential coefficients will be met with later on. They are like this:

$\frac{d^2y}{dx^2}$; which is read “*dee-two-wy over dee-eks-squared,*”

and it means that the operation of differentiating y with respect to x has been (or has to be) performed twice over.

Another way of indicating that a function has been differentiated is by putting an accent to the symbol of the function. Thus if $y = F(x)$, which means that y is some unspecified function of x (see p. 8), we may write $F'(x)$ instead of $\frac{d(F(x))}{dx}$. Similarly, $F''(x)$ will mean that the original function $F(x)$ has been differentiated twice over with respect to x .

Simplest Cases

Now let us see how, on first principles, we can differentiate some simple algebraical expression.

Case 1.

Let us begin with the simple expression $y = x^2$. Now remember that the fundamental notion about the calculus is the idea of *growing*. Mathematicians call it *varying*. Now as y and x^2 are equal to one another, it is clear that if x grows, x^2 will also grow. And if x^2 grows, then y will also grow. What we have got to find out is the proportion between the growing of y and the growing of x . In other words our task is to find out the ratio between dy and dx , or, in brief, to find the value of $\frac{dy}{dx}$.

Let x , then, grow a little bit bigger and become $x + dx$; similarly, y will grow a bit bigger and will become $y + dy$. Then, clearly, it will still be true that the enlarged y will be equal to the square of the enlarged x . Writing this down, we have:

$$y + dy = (x + dx)^2.$$

Doing the squaring we get:

$$y + dy = x^2 + 2x \cdot dx + (dx)^2. \quad 4.1$$

What does $(dx)^2$ mean? Remember that dx meant a bit—a little bit—of x . Then $(dx)^2$ will mean a little bit of a little bit of x ; that is, as explained above (p. 3), it is a small quantity of the second order of smallness. It may therefore be discarded as quite inconsiderable in comparison with the other terms. Leaving it out, we then have:

$$y + dy = x^2 + 2x \cdot dx.$$

Now $y = x^2$; so let us subtract this from the equation and we have left

$$dy = 2x \cdot dx.$$

Dividing across by dx , we find

$$4.2 \quad \frac{dy}{dx} = 2x.$$

Now *this** is what we set out to find. The ratio of the growing of y to the growing of x is, in the case before us, found to be $2x$.

Numerical example.

Suppose $x = 100$ and $\therefore y = 10,000$. Then let x grow till it becomes 101 (that is, let $dx = 1$). Then the enlarged y will be $101 \times 101 = 10,201$. But if we agree that we may ignore small quantities of the second order, 1 may be rejected as compared with 10,000; so we may round off the enlarged y to 10,200. y has grown from 10,000 to 10,200; the bit added on is dy , which is therefore 200.

$\frac{dy}{dx} = \frac{200}{1} = 200$. According to the algebra-working of the previous paragraph, we find $\frac{dy}{dx} = 2x$. And so it is; for $x = 100$ and $2x = 200$.

But, you will say, we neglected a whole unit.

Well, try again, making dx a still smaller bit.

Try $dx = \frac{1}{10}$. Then $x + dx = 100.1$, and

$$4.3 \quad (x + dx)^2 = 100.1 \times 100.1 = 10,020.01.$$

Now the last figure 1 is only one-millionth part of the 10,000, and is utterly negligible; so we may take 10,020 without the little decimal at the end. And this makes $dy = 20$; and $\frac{dy}{dx} = \frac{20}{0.1} = 200$, which is still the same as $2x$.

Case 2.

Try differentiating $y = x^3$ in the same way.

* N.B.—This ratio $\frac{dy}{dx}$ is the result of differentiating y with respect to x . Differentiating means finding the differential coefficient. Suppose we had some other function of x , as, for example, $u = 7x^2 + 3$. Then if we were told to differentiate this with respect to x , we should have to find $\frac{du}{dx}$, or, what is the same thing, $\frac{d(7x^2+3)}{dx}$. On the other hand, we may have a case in which time was the independent variable (see p. 9), such as this: $y = b + \frac{1}{2}at^2$. Then, if we were told to differentiate it, that means we must find its differential coefficient with respect to t . So that then our business would be to try to find $\frac{dy}{dt}$, that is, to find $\frac{d(b+\frac{1}{2}at^2)}{dt}$.

We let y grow to $y + dy$, while x grows to $x + dx$.

Then we have

$$y + dy = (x + dx)^3. \quad 4.4$$

Doing the cubing we obtain

$$y + dy = x^3 + 3x^2 \cdot dx + 3x(dx)^2 + (dx)^3. \quad 4.5$$

Now we know that we may neglect small quantities of the second and third orders; since, when dy and dx are both made indefinitely small, $(dx)^2$ and $(dx)^3$ will become indefinitely smaller by comparison. So, regarding them as negligible, we have left:

$$y + dy = x^3 + 3x^2 \cdot dx. \quad 4.6$$

But $y = x^3$; and, subtracting this, we have:

$$dy = 3x^2 \cdot dx,$$

$$\text{and } \frac{dy}{dx} = 3x^2. \quad 4.7$$

Case 3.

Try differentiating $y = x^4$. Starting as before by letting both y and x grow a bit, we have:

$$y + dy = (x + dx)^4.$$

Working out the raising to the fourth power, we get

$$y + dy = x^4 + 4x^3 dx + 6x^2(dx)^2 + 4x(dx)^3 + (dx)^4.$$

Then striking out the terms containing all the higher powers of dx , as being negligible by comparison, we have

$$y + dy = x^4 + 4x^3 dx.$$

Subtracting the original $y = x^4$, we have left

$$dy = 4x^3 dx,$$

$$\text{and } \frac{dy}{dx} = 4x^3. \quad 4.8$$

Now all these cases are quite easy. Let us collect the results to see if we can infer any general rule. Put them in two columns, the values of y in one and the corresponding values found for $\frac{dy}{dx}$ in the other: thus

y	$\frac{dy}{dx}$
x^2	$2x$
x^3	$3x^2$
x^4	$4x^3$

Just look at these results: the operation of differentiating appears to have had the effect of diminishing the power of x by 1 (for example in the last case reducing x^4 to x^3), and at the same time multiplying by a number (the same number in fact which originally appeared as the power). Now, when you have once seen this, you might easily conjecture how the others will run. You would expect that differentiating x^5 would give $5x^4$, or differentiating x^6 would give $6x^5$. If you hesitate, try one of these, and see whether the conjecture comes right.

Try $y = x^5$.

$$\begin{aligned}\text{Then } y + dy &= (x + dx)^5 \\ &= x^5 + 5x^4 dx + 10x^3(dx)^2 + 10x^2(dx)^3 \\ &\quad + 5x(dx)^4 + (dx)^5.\end{aligned}$$

4.9

Neglecting all the terms containing small quantities of the higher orders, we have left

$$\begin{aligned}y + dy &= x^5 + 5x^4 dx, \\ \text{and subtracting } y = x^5 &\text{ leaves us} \\ dy &= 5x^4 dx,\end{aligned}$$

4.10

$$\text{whence } \frac{dy}{dx} = 5x^4, \text{ exactly as we supposed.}$$

Following out logically our observation, we should conclude that if we want to deal with any higher power,—call it n —we could tackle it in the same way.

$$\text{Let } y = x^n,$$

then, we should expect to find that

$$\frac{dy}{dx} = nx^{(n-1)}.$$

4.11

For example, let $n = 8$, then $y = x^8$; and differentiating it would give $\frac{dy}{dx} = 8x^7$.

And, indeed, the rule that differentiating x^n gives as the result nx^{n-1} is true for all cases where n is a whole number and positive. [Expanding $(x + dx)^n$ by the binomial theorem will at once show this.] But the question whether it is true for cases where n has negative or fractional values requires further consideration.

Case of a negative power.

Let $y = x^{-2}$. Then proceed as before:

$$\begin{aligned} y + dy &= (x + dx)^{-2} \\ &= x^{-2} \left(1 + \frac{dx}{x} \right)^{-2}. \end{aligned} \quad 4.12$$

Expanding this by the binomial theorem (see p. 141), we get

$$\begin{aligned} &= x^{-2} \left[1 - \frac{2dx}{x} + \frac{2(2+1)}{1 \times 2} \left(\frac{dx}{x} \right)^2 - \text{etc.} \right] \\ &= x^{-2} - 2x^{-3} \cdot dx + 3x^{-4}(dx)^2 - 4x^{-5}(dx)^3 + \text{etc.} \end{aligned}$$

So, neglecting the small quantities of higher orders of smallness, we have:

$$y + dy = x^{-2} - 2x^{-3} \cdot dx.$$

Subtracting the original $y = x^{-2}$, we find

$$\begin{aligned} dy &= -2x^{-3}dx, \\ \frac{dy}{dx} &= -2x^{-3}. \end{aligned} \quad 4.13$$

And this is still in accordance with the rule inferred above.

Case of a fractional power.

Let $y = x^{\frac{1}{2}}$. Then, as before,

$$\begin{aligned} y + dy &= (x + dx)^{\frac{1}{2}} = x^{\frac{1}{2}} \left(1 + \frac{dx}{x} \right)^{\frac{1}{2}} \\ &= \sqrt{x} + \frac{1}{2} \frac{dx}{\sqrt{x}} - \frac{1}{8} \frac{(dx)^2}{x\sqrt{x}} + \text{terms with higher} \\ &\quad \text{powers of } dx. \end{aligned} \quad 4.14$$

Subtracting the original $y = x^{\frac{1}{2}}$, and neglecting higher powers we have left:

$$4.15 \quad dy = \frac{1}{2} \frac{dx}{\sqrt{x}} = \frac{1}{2} x^{-\frac{1}{2}} \cdot dx,$$

and $\frac{dy}{dx} = \frac{1}{2} x^{-\frac{1}{2}}$. Agreeing with the general rule.

Summary. Let us see how far we have got. We have arrived at the following rule: To differentiate x^n , multiply by the power and reduce the power by one, so giving us nx^{n-1} as the result.

Exercises I. (See p. 24 for Answers.)

Differentiate the following:

(1) $y = x^{13}$

(2) $y = x^{-\frac{3}{2}}$

(3) $y = x^{2a}$

(4) $u = t^{2.4}$

(5) $z = \sqrt[3]{u}$

(6) $y = \sqrt[3]{x^{-5}}$

(7) $u = \sqrt[5]{\frac{1}{x^8}}$

(8) $y = 2x^a$

(9) $y = \sqrt[4]{x^3}$

(10) $y = \sqrt[n]{\frac{1}{x^m}}$



You have now learned how to differentiate powers of x . How easy it is!

Next Stage. What to do with Constants

IN our equations we have regarded x as growing, and as a result of x being made to grow y also changed its value and grew. We usually think of x as a quantity that we can vary; and, regarding the variation of x as a sort of *cause*, we consider the resulting variation of y as an *effect*. In other words, we regard the value of y as depending on that of x . Both x and y are variables, but x is the one that we operate upon, and y is the “dependent variable.” In all the preceding chapter we have been trying to find out rules for the proportion which the dependent variation in y bears to the variation independently made in x .

Our next step is to find out what effect on the process of differentiating is caused by the presence of *constants*, that is, of numbers which don’t change when x or y change their values.

Added Constants.

Let us begin with some simple case of an added constant, thus:

$$\text{Let } y = x^3 + 5.$$

5.1

Just as before, let us suppose x to grow to $x + dx$ and y to grow to $y + dy$.

$$\begin{aligned} \text{Then: } y + dy &= (x + dx)^3 + 5 \\ &= x^3 + 3x^2 dx + 3x(dx)^2 + (dx)^3 + 5. \end{aligned}$$

Neglecting the small quantities of higher orders, this becomes

$$y + dy = x^3 + 3x^2 \cdot dx + 5.$$

Subtract the original $y = x^3 + 5$, and we have left:

$$dy = 3x^2 dx.$$

5.2

$$\frac{dy}{dx} = 3x^2.$$

So the 5 has quite disappeared. It added nothing to the growth of x , and does not enter into the differential coefficient. If we had put 7, or 700, or any other number, instead of 5, it would have disappeared. So if we take the letter a , or b , or c to represent any constant, it will simply disappear when we differentiate.

If the additional constant had been of negative value, such as -5 or $-b$, it would equally have disappeared.

Multiplied Constants.

Take as a simple experiment this case:

$$\text{Let } y = 7x^2.$$

Then on proceeding as before we get:

$$\begin{aligned} y + dy &= 7(x + dx)^2 \\ &= 7\{x^2 + 2x \cdot dx + (dx)^2\} \\ &= 7x^2 + 14x \cdot dx + 7(dx)^2. \end{aligned}$$

Then, subtracting the original $y = 7x^2$, and neglecting the last term, we have

$$dy = 14x \cdot dx.$$

5.3

$$\frac{dy}{dx} = 14x.$$

Let us illustrate this example by working out the graphs of the equations $y = 7x^2$ and $\frac{dy}{dx} = 14x$, by assigning to x a set of successive values, 0, 1, 2, 3, etc., and finding the corresponding values of y and of $\frac{dy}{dx}$.

These values we tabulate as follows:

x	0	1	2	3	4	5	-1	-2	-3
y	0	7	28	63	112	175	7	28	63
$\frac{dy}{dx}$	0	14	28	42	56	70			

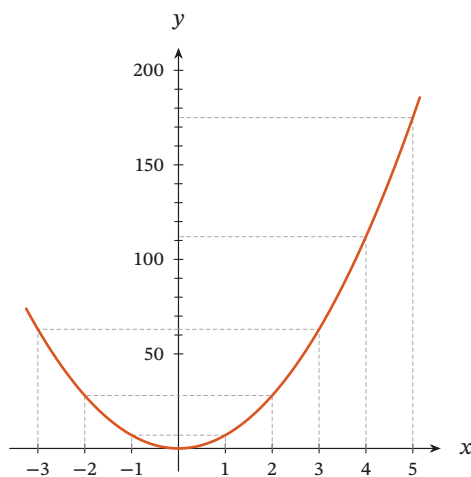


Fig. 6

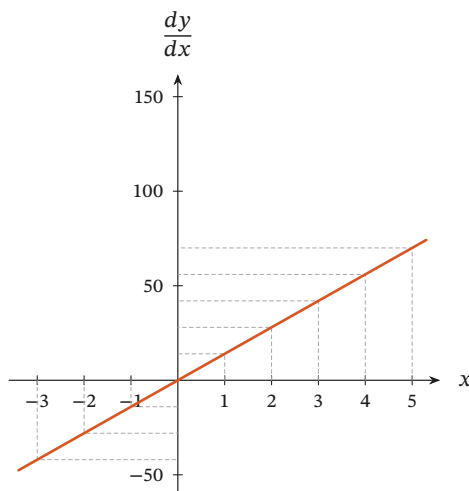


Fig. 6a

Now plot these values to some convenient scale, and we obtain the two curves, Figs. 6 and 6a.

Carefully compare the two figures, and verify by inspection that the height of the ordinate of the derived curve, Fig. 6a, is proportional to the *slope* of the original curve,* Fig. 6, at the corresponding value of x . To the left of the origin, where the original curve slopes negatively (that is, downward from left to right) the corresponding ordinates of the derived curve are negative.

* See p. 77 about *slopes* of curves.

Now if we look back at p. 11, we shall see that simply differentiating x^2 gives us $2x$. So that the differential coefficient of $7x^2$ is just 7 times as big as that of x^2 . If we had taken $8x^2$, the differential coefficient would have come out eight times as great as that of x^2 . If we put $y = ax^2$, we shall get

$$\frac{dy}{dx} = a \times 2x.$$

5.4

If we had begun with $y = ax^n$, we should have had $\frac{dy}{dx} = a \times nx^{n-1}$. So that any mere multiplication by a constant reappears as a mere multiplication when the thing is differentiated. And, what is true about multiplication is equally true about *division*: for if, in the example above, we had taken as the constant $\frac{1}{7}$ instead of 7, we should have had the same $\frac{1}{7}$ come out in the result after differentiation.

Some Further Examples.

The following further examples, fully worked out, will enable you to master completely the process of differentiation as applied to ordinary algebraical expressions, and enable you to work out by yourself the examples given at the end of this chapter.

(1) Differentiate $y = \frac{x^5}{7} - \frac{3}{5}$.

$\frac{3}{5}$ is an added constant and vanishes (see p. 17).

We may then write at once

$$\frac{dy}{dx} = \frac{1}{7} \times 5 \times x^{5-1},$$

5.5 or $\frac{dy}{dx} = \frac{5}{7}x^4$.

(2) Differentiate $y = a\sqrt{x} - \frac{1}{2}\sqrt{a}$.

The term $\frac{1}{2}\sqrt{a}$ vanishes, being an added constant; and as $a\sqrt{x}$, in the index form, is written $ax^{\frac{1}{2}}$, we have

$$\frac{dy}{dx} = a \times \frac{1}{2} \times x^{\frac{1}{2}-1} = \frac{a}{2} \times x^{-\frac{1}{2}},$$

5.6 or $\frac{dy}{dx} = \frac{a}{2\sqrt{x}}$.

(3) If $ay + bx = by - ax + (x + y)\sqrt{a^2 - b^2}$,

find the differential coefficient of y with respect to x .

As a rule an expression of this kind will need a little more knowledge than we have acquired so far; it is, however, always worth while to try whether the expression can be put in a simpler form.

First we must try to bring it into the form $y =$ some expression involving x only.

The expression may be written

5.7 $(a - b)y + (a + b)x = (x + y)\sqrt{a^2 - b^2}$.

Squaring, we get

5.8
$$\begin{aligned} (a - b)^2y^2 + (a + b)^2x^2 + 2(a + b)(a - b)xy \\ = (x^2 + y^2 + 2xy)(a^2 - b^2), \end{aligned}$$

which simplifies to

$$\begin{aligned} & (a-b)^2y^2 + (a+b)^2x^2 = x^2(a^2-b^2) + y^2(a^2-b^2); \\ \text{or} \quad & [(a-b)^2 - (a^2-b^2)]y^2 = [(a^2-b^2) - (a+b)^2]x^2, \\ \text{that is} \quad & 2b(b-a)y^2 = -2b(b+a)x^2; \end{aligned}$$

$$\text{hence} \quad y = \sqrt{\frac{a+b}{a-b}} x$$

$$\text{and} \quad \frac{dy}{dx} = \sqrt{\frac{a+b}{a-b}}. \quad 5.9$$

(4) The volume of a cylinder of radius r and height h is given by the formula $V = \pi r^2 h$. Find the rate of variation of volume with the radius when $r = 5.5$ in. and $h = 20$ in. If $r = h$, find the dimensions of the cylinder so that a change of 1 in. in radius causes a change of 400 cub. in. in the volume.

The rate of variation of V with regard to r is

$$\frac{dV}{dr} = 2\pi rh. \quad 5.10$$

If $r = 5.5$ in. and $h = 20$ in. this becomes 690.8. It means that a change of radius of 1 inch will cause a change of volume of 690.8 cub. inch. This can be easily verified, for the volumes with $r = 5$ and $r = 6$ are 1570 cub. in. and 2260.8 cub. in. respectively, and $2260.8 - 1570 = 690.8$.

Also, if

$$r = h, \quad \frac{dV}{dr} = 2\pi r^2 = 400 \quad \text{and} \quad r = h = \sqrt{\frac{400}{2\pi}} = 7.98 \text{ in.} \quad 5.11$$

(5) The reading θ of a Féry's Radiation pyrometer is related to the Centigrade temperature t of the observed body by the relation

$$\frac{\theta}{\theta_1} = \left(\frac{t}{t_1}\right)^4, \quad 5.12$$

where θ_1 is the reading corresponding to a known temperature t_1 of the observed body.

Compare the sensitiveness of the pyrometer at temperatures 800° C., 1000° C., 1200° C., given that it read 25 when the temperature was 1000° C.

The sensitiveness is the rate of variation of the reading with the temperature, that is $\frac{d\theta}{dt}$. The formula may be written

$$5.13 \quad \theta = \frac{\theta_1}{t_1^4} t^4 = \frac{25t^4}{1000^4},$$

and we have

$$5.14 \quad \frac{d\theta}{dt} = \frac{100t^3}{1000^4} = \frac{t^3}{10,000,000,000}.$$

When $t = 800$, 1000 and 1200 , we get $\frac{d\theta}{dt} = 0.0512$, 0.1 and 0.1728 respectively.

The sensitiveness is approximately doubled from 800° to 1000° , and becomes three-quarters as great again up to 1200° .

Exercises II. (See p. 25 for Answers.)

Differentiate the following:

$$(1) \quad y = ax^3 + 6.$$

$$(2) \quad y = 13x^{\frac{3}{2}} - c.$$

$$(3) \quad y = 12x^{\frac{1}{2}} + c^{\frac{1}{2}}.$$

$$(4) \quad y = c^{\frac{1}{2}} x^{\frac{1}{2}}.$$

$$(5) \quad u = \frac{az^n - 1}{c}.$$

$$(6) \quad y = 1.18t^2 + 22.4.$$

Make up some other examples for yourself, and try your hand at differentiating them.

(7) If l_t and l_0 be the lengths of a rod of iron at the temperatures t° C. and 0° C. respectively, then $l_t = l_0(1 + 0.000012t)$. Find the change of length of the rod per degree Centigrade.

(8) It has been found that if c be the candle power of an incandescent electric lamp, and V be the voltage, $c = aV^b$, where a and b are constants.

Find the rate of change of the candle power with the voltage, and calculate the change of candle power per volt at 80, 100 and 120 volts in the case of a lamp for which $a = 0.5 \times 10^{-10}$ and $b = 6$.

(9) The frequency n of vibration of a string of diameter D , length L and specific gravity σ , stretched with a force T , is given by

$$n = \frac{1}{DL} \sqrt{\frac{gT}{\pi\sigma}}.$$

Find the rate of change of the frequency when D , L , σ and T are varied singly.

- (10) The greatest external pressure P which a tube can support without collapsing is given by

$$P = \left(\frac{2E}{1 - \sigma^2} \right) \frac{t^3}{D^3},$$

where E and σ are constants, t is the thickness of the tube and D is its diameter. (This formula assumes that $4t$ is small compared to D .)

Compare the rate at which P varies for a small change of thickness and for a small change of diameter taking place separately.

- (11) Find, from first principles, the rate at which the following vary with respect to a change in radius:
- (a) the circumference of a circle of radius r ;
 - (b) the area of a circle of radius r ;
 - (c) the lateral area of a cone of slant dimension l ;
 - (d) the volume of a cone of radius r and height h ;
 - (e) the area of a sphere of radius r ;
 - (f) the volume of a sphere of radius r .
- (12) The length L of an iron rod at the temperature T being given by $L = l_t[1 + 0.000012(T - t)]$, where l_t is the length at the temperature t , find the rate of variation of the diameter D of an iron tyre suitable for being shrunk on a wheel, when the temperature T varies.

Answers to Exercises

Exercises I. (p. 16.)

$$(1) \frac{dy}{dx} = 13x^{12}. \quad (2) \frac{dy}{dx} = -\frac{3}{2}x^{-\frac{5}{2}}. \quad (3) \frac{dy}{dx} = 2ax^{(2a-1)}.$$

$$(4) \frac{du}{dt} = 2.4t^{1.4}. \quad (5) \frac{dz}{du} = \frac{1}{3}u^{-\frac{2}{3}}. \quad (6) \frac{dy}{dx} = -\frac{5}{3}x^{-\frac{8}{3}}.$$

$$(7) \frac{du}{dx} = -\frac{8}{5}x^{-\frac{13}{5}}. \quad (8) \frac{dy}{dx} = 2ax^{a-1}.$$

$$(9) \frac{dy}{dx} = \frac{3}{q}x^{\frac{3-q}{q}}. \quad (10) \frac{dy}{dx} = -\frac{m}{n}x^{-\frac{m+n}{n}}.$$

Exercises II. (p. 22.)

$$(1) \frac{dy}{dx} = 3ax^2. \quad (2) \frac{dy}{dx} = 13 \times \frac{3}{2}x^{\frac{1}{2}}. \quad (3) \frac{dy}{dx} = 6x^{-\frac{1}{2}}.$$

$$(4) \frac{dy}{dx} = \frac{1}{2}c^{\frac{1}{2}}x^{-\frac{1}{2}}. \quad (5) \frac{du}{dz} = \frac{an}{c}z^{n-1}. \quad (6) \frac{dy}{dt} = 2.36t.$$

$$(7) \frac{dl_t}{dt} = 0.000012 \times l_0.$$

$$(8) \frac{dC}{dV} = abV^{b-1}, 0.98, 3.00 \text{ and } 7.47 \text{ candle power per volt respectively.}$$

$$(9) \frac{dn}{dD} = -\frac{1}{LD^2}\sqrt{\frac{gT}{\pi\sigma}}, \quad \frac{dn}{dL} = -\frac{1}{DL^2}\sqrt{\frac{gT}{\pi\sigma}},$$

$$\frac{dn}{d\sigma} = -\frac{1}{2DL}\sqrt{\frac{gT}{\pi\sigma^3}}, \quad \frac{dn}{dT} = \frac{1}{2DL}\sqrt{\frac{g}{\pi\sigma T}}.$$

$$(10) \frac{\text{Rate of change of } P \text{ when } t \text{ varies}}{\text{Rate of change of } P \text{ when } D \text{ varies}} = -\frac{D}{t}.$$

$$(11) 2\pi, 2\pi r, \pi l, \frac{2}{3}\pi rh, 8\pi r, 4\pi r^2.$$

$$(12) \frac{dD}{dT} = \frac{0.000012l_t}{\pi}.$$